Advanced Mechanics: Final Exam

Semester Ib 2022-2023

January 24th, 2023

Instructions

1. You are not allowed to use the book or the lecture notes, nor other notes or books.
2. You have 2 hours to complete the exam.
3. Please write your student number on each paper sheet you hand in.
4. Please use a separate paper sheet for each problem.
5. Please raise your hand for more paper or to ask a question.
6. Some useful equations are provided on the next page.
7. It is important that you show your work.

Points for Each Problem

- Problem 1: 15 points
- Problem 2: 30 points
- Problem 3: 20 points
- Problem 4: 35 points

Useful Equations

Principal Moments of Inertia

$$I_x = \int dm(y^2 + z^2) \tag{0.1}$$

$$I_y = \int dm(x^2 + z^2) \tag{0.2}$$

$$I_z = \int dm(x^2 + y^2) \tag{0.3}$$

Products of Inertia

$$I_{xy} = -\int dm\, xy \tag{0.4a}$$

$$I_{xz} = -\int dm\, xz \tag{0.4b}$$

$$I_{yz} = -\int dm\, yz \tag{0.4c}$$

Moment of Inertia of a Uniform Cylinder

For a uniform cylinder of mass M and radius R with respect to its axis:

$$I = \frac{1}{2}MR^2 \tag{0.5}$$

Center of Mass for N Point Masses m_i

$$\vec{r}_{\text{CM}} = \frac{1}{M}\sum_{i=1}^{N} m_i\vec{r}_i \quad \text{where } M = \sum_{i=1}^{N} m_i$$

Problem 1

Find the principal moments of inertia for a book of uniformly distributed mass M and sides (a, b, c) as shown in Figure 1, rotating about its center. Is any of the products of inertia non-zero? [**15 points**]

Problem 2

Consider two point masses m_1 and m_2 moving in a plane (assume no friction) and interacting with each other by means of a potential $V(r) = \frac{1}{2}kr^2$, where r is the distance between them (see Figure 2).

(a) Starting from the Lagrangian for the two point masses in terms of their respective coordinates $\vec{r}_1$ and $\vec{r}_2$, obtain new coordinates in terms of the center of mass $\vec{R}$ and relative position $\vec{r}$. [**10 points**]

(b) Derive the equations of motion for the coordinates $\vec{R} = (X, Y)$ and $\vec{r} = (x, y)$. What is the frequency of the relative motion? [**10 points**]

(c) Compute the generalized momenta and from there obtain the Hamiltonian. Then verify explicitly that the Hamiltonian is equal to the total energy. [**10 points**]

Problem 3

Consider the object in Figure 3: a massless string is suspended vertically from a fixed point and wrapped around a uniform cylinder of mass M and radius R. The cylinder moves vertically down due to gravity, rotating as the string unwinds. [The moment of inertia of the cylinder is provided in the formula sheet.]

(a) Derive the Lagrangian, using y as your generalized coordinate. [**5 points**]

(b) Obtain the equation of motion and verify that the cylinder accelerates downwards with acceleration equal to $2g/3$. [**5 points**]

Problem 4

Consider the system represented in Fig. 4, consisting of two point masses, m_1 and m_2, connected by an inextensible string passing over a frictionless massless pulley. The first mass is free to move on a frictionless horizontal table, while the second mass moves vertically. Use coordinates x and y equal to the distances of, respectively, m_1 and m_2 from the pulley.

(4a)

Compute the Lagrangian in x and y coordinates.
[**10 points**]

(4b)

Use the modified Lagrange equations to derive the generalised force.
[**15 points**]

(4c)

Using the Newtonian approach, verify that the generalised force derived in (4b) is equal to the tension of the string on the two masses.
[**10 points**]

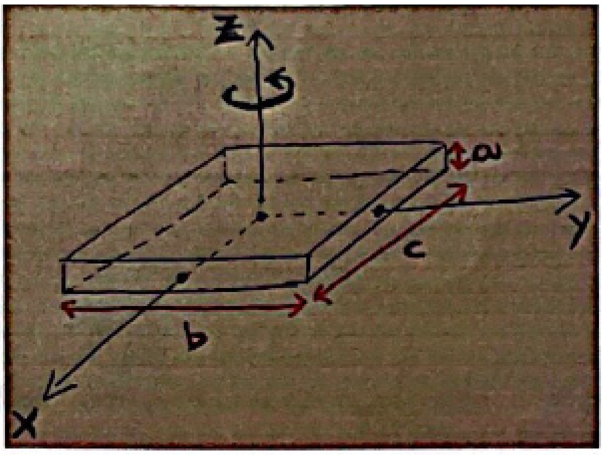

Figure 1:

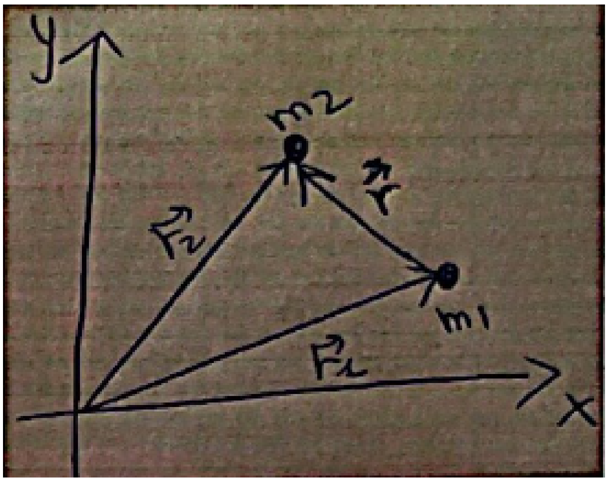

Figure 2:

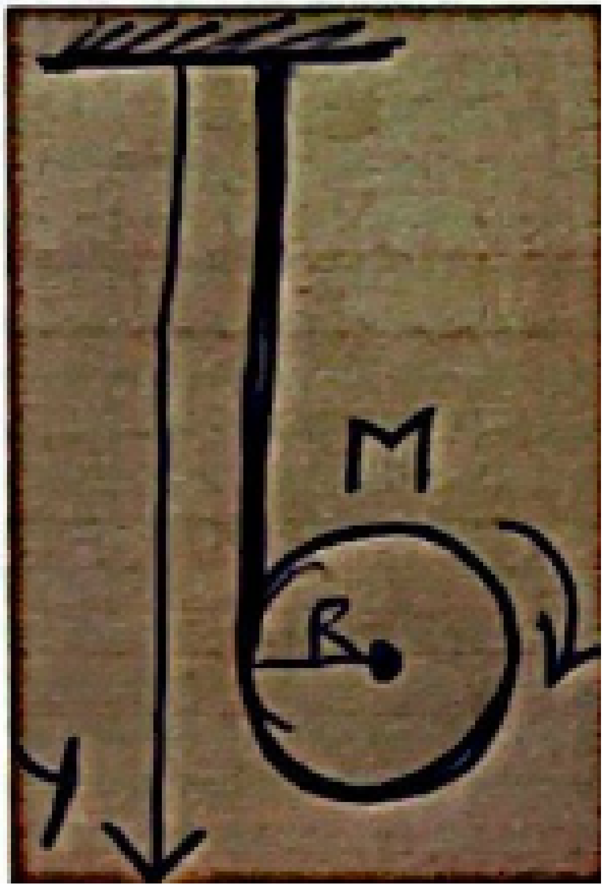

Figure 3:

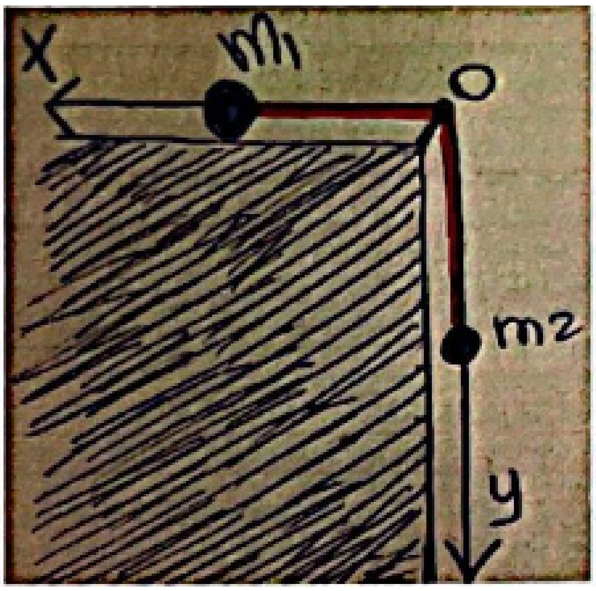

Figure 4: